Part I. The 2D Classical Coulomb Gas near the Zero-Density Kosterlitz–Thouless Critical Point: Correlations and Critical Line

The 2D classical Coulomb gas undergoes the famous Kosterlitz–Thouless (KT) transition between a high-temperature conducting phase and a low-temperature insulating phase. We present various studies of the correlations in the insulating phase near the zero-density critical point. First, we briefly recall the phenomenological approach of Kosterlitz and Thouless. This theory predicts that the decay of the charge correlation is entirely controlled by the bare Coulomb potential between opposite charges only renormalized by the dielectric constant ε. Then, we present an analysis of the low-fugacity expansions of the correlations. The particle correlations are found to decay as $1/r^4$. The large-distance decay of the charge correlation is shown to be tightly related to the behavior of $1/\varepsilon$ in the regime of interest. Systematic resummations allow one to recover the algebraic decay predicted by the heuristic KT model. This settles on a rigorous basis various assumptions of this model. In particular, the nested pair mechanism naturally arises in the resummation scheme. Finally, we describe the phase diagram of the system according to the most recent calculations which include finite-density effects.

KEY WORDS: Coulomb gas; Kosterlitz–Thouless transition; correlations; fugacity expansions; critical line.

1. INTRODUCTION

The Coulomb Gas is a charge-symmetric two-component plasma with charges e and $-e$, which interact through the two-body logarithmic potential, defined as the solution of the Poisson equation in 2 dimension,

$$\Delta v_c(\mathbf{r}) = -2\pi\delta(\mathbf{r}) \tag{1.1}$$

namely $v_c(r) = -\ln(r/L)$ (L is an irrelevant scale length which fixes the zero of the potential.) When the temperature is sufficiently high, the system of point charges is well-behaved.[1] The scaling invariance of the logarithmic potential implies that the excess properties of the system depend only on the dimensionless coupling constant $\Gamma = \beta e^2$. Moreover, the equation of state is then exactly known.[1,2] However, because of the short-ranged attraction between opposite charges, this system collapses when $\Gamma = 2$. Subsequently, for lower temperatures, a system of charged hard disks with diameter σ is to be considered. (The interaction between two disks is infinite, if the distance is smaller than σ, and it reduces to the Coulomb interaction at larger distances). Then the state of the system is characterized by two dimensionless parameters, Γ and $\rho\sigma^2$, where ρ is the total particle density $\rho = \rho_+ + \rho_-$. (In the following L is chosen to be equal to σ.)

On the other hand, because of the long range of the confining logarithmic interaction, the system undergoes the famous Kosterlitz–Thouless (KT) transition[3,4] between a high-temperature conducting phase, where the dielectric constant ε is infinite, and a low-temperature dielectric phase, where ε has a finite value. The theorem of Mermin and Wagner[5] shows that, in two-dimensional systems with continuous symmetries, any order parameter vanishes in zero-field. (For instance, there is no spontaneous magnetization in a 2D XY magnet.) In fact, the KT transition is characterized by a singularity in the response to an external excitation. Moreover the transition is of infinite order. The KT transition is of great interest, since it is the archetype of a universality class of 2D transitions induced by a condensation of topological excitations (such as defects in quasicrystals, vortices in He^4 films). For the present system of charged hard disks, at low density, the conducting and dielectric phases are separated by a line of critical points which ends at $\Gamma = 4$ for $\rho = 0$. (This zero-density critical value remains unchanged if one species of particles is fixed on a lattice.[6]) At higher density, this critical line bifurcates into a first order liquid-gas coexistence curve.[7,8,9]

From the microscopic point of view, the effective potential between infinitesimal external charges decays exponentially in the conducting phase, whereas, in the dielectric phase, this effective potential is proportional to the bare logarithmic potential, as shown rigorously by Fröhlich and Spencer in 1981.[10] By definition, the renormalized multiplicative constant is equal to $1/\varepsilon$ which is related to the internal correlations through the linear response theory,

$$\frac{1}{\varepsilon} = 1 + \frac{\pi\beta}{2} \int d\mathbf{r}\, r^2 C(r). \tag{1.2}$$

In (1.2), $C(r)$ is the internal charge correlation,

$$C(\mathbf{r}) = \langle Q(\mathbf{r})\, Q(\mathbf{0}) \rangle \tag{1.3}$$

where $Q(\mathbf{r})$ is the total microscopic charge density at point $\mathbf{r}$ and $\langle ... \rangle$ denotes a thermal equilibrium average. In terms of the internal particle correlations, namely the two-body truncated distribution functions $\rho^T_{++} = \rho^T_{--}$ and $\rho^T_{+-} = \rho^T_{-+}$, $C(\mathbf{r})$ reads

$$C(\mathbf{r}) = e^2\{2[\rho^T_{++}(r) - \rho^T_{+-}(r)] + \rho\delta(\mathbf{r})\}. \tag{1.4}$$

In the conducting phase, the internal correlations decay exponentially, as in the meanfield Debye–Hückel theory.[11] This was rigorously shown in the high-temperature limit,[12] and it was also explicitly checked in solvable models for $\Gamma = 2$.[13, 14] On the other hand, Martin and Gruber[15] have shown that, if the two-body correlations decay faster than $1/r^4$ and if the three- and four-body correlations decay faster than $1/r^3$, then $1/\varepsilon$ is zero. Thus, in the dielectric phase where $1/\varepsilon$ is finite, some particle correlations must fall off algebraically. However, the exact structure of this algebraic tail has been an open question for a long time.

In this brief review, we describe various approaches dealing with the large-distance behavior of the internal correlations in the dielectric phase. First, in Section 2, we recall the phenomenological approach by Kosterlitz and Thouless,[3] which is based on an iterated mean-field theory for the charge correlation. Their picture of nested pairs leads to a model of independent pairs embedded in a polarizable continuous medium. Then, as expected, $C(r)$ is found to decay algebraically, namely as $1/r^{\Gamma/\varepsilon}$. However, in this analysis, various crucial assumptions remain to be settled from the first principles of Statistical Mechanics. Moreover, the method does not provide any information about the particle correlations. In Section 3, we present an analysis of the latter correlations starting from expansions with respect to the fugacity z.[16] The large-distance behavior of the particle correlations is shown to be controlled by the fluctuations of dipolar potentials. Thus, ρ^T_{++} and ρ^T_{+-} decay as $1/r^4$ with the same coefficient, so that $C(r)$ falls off faster. Near the zero-density critical point, where both $\Gamma - 4$ and z are small parameters, systematic resummations for $C(r)$ at all orders in z exhibit the nested-pair mechanism introduced heuristically by Kosterlitz and Thouless. The leading term in the large-r expansion of $C(r)$ does coincide with the mean-field prediction. Furthermore, an infinite set of algebraic subleading terms contributes to the behaviour of $1/\varepsilon$ near the zero-density critical point. Finally, in Section 4, we present some recent results about the phase diagram, and we briefly discuss some open questions.

2. THE KT PHENOMENOLOGICAL APPROACH

2.1. The KT Model

The first coherent model which described the dielectric phase was a phenomenological approach by Kosterlitz and Thouless.[3] These authors supposed that, in the dielectric phase, the charges form some kind of pairs, because of the confining logarithmic potential, and they assumed that a given pair is screened only by pairs which are smaller. For instance, in

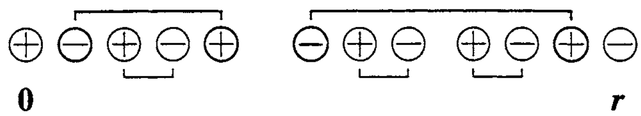

Fig. 1. The pair ($\oplus\mathbf{0}$, $\ominus\mathbf{r}$) screened by nested smaller ones.

Fig. 1, the largest pair is screened directly by two pairs, which are themselves screened by smaller ones. The charge correlation $C(r)$ is supposed to be entirely controlled by such configurations. The physical picture that emerges from the latter configurations is that of independent neutral pairs embedded in a polarizable continuous medium in which the dielectric constant $\varepsilon(r)$ depends on the distance r between the charges of the pair. This picture is also called an iterated mean-field theory.

More precisely, by analogy with the result of the linear response theory (1.2), the dielectric constant $\varepsilon(r)$ is related to a truncated second-moment of the charge correlation, where only neutral pairs smaller than the distance r contribute so that

$$\frac{1}{\varepsilon(r)} = 1 + \frac{\pi\beta}{2} \int_{\sigma < r' < r} d\mathbf{r}'\, r'^2 C(r') \tag{2.1}$$

Notice that $\varepsilon(\infty)$ is nothing but the macroscopic dielectic constant given by (1.2). On the other hand, the charge correlation is calculated in the independent neutral pair approximation,

$$C(r) = -2e^2 \frac{z^2}{\sigma^4} \exp\left[-\int_{\sigma < r' < r} dr' \frac{\Gamma}{\varepsilon(r')\, r'} \right] \tag{2.2}$$

In the Boltzmann factor, the effective potential of a pair with size r is equal to the work which is necessary to form the pair in a polarizable medium with dielectric constant $\varepsilon(r)$.

The coupled equations (2.1)–(2.2) allow one to describe the famous KT transition.[17] They are equivalent to the flow equations of the Renormalization Group approach developed by Kosterlitz for the Coulomb gas[4] and by Amit, Goldschmidt and Grinstein for the equivalent Sine–Gordon model in field theory.[18] In particular, these equations lead to the well-known equation of the critical line,

$$\Gamma_{KT}(z) - 4 = 8\pi z \tag{2.3}$$

Moreover, we mention that there exists another phenomenological approach based on some kind of chemical picture.[19] In this approach, the system is viewed as a mixture of free charges and neutral dipoles, and the correlations are determined from a closed set of integral equations. The results for the KT, transition are shown to be equivalent to those derived from (2.1) and (2.2), while the singular nature of the correlations on the critical line is also investigated.

2.2. What About First Principles?

However, the validity of the basic assumptions in the KT phenomenological approach is to be proved. Indeed, several questions may be asked. First, why is one pair screened only by smaller ones? (Note that the integral in (2.1) is restricted to $r' < r$.) Second, what about the correlations between charges with the same sign, which appear in the definition of the charge correlation and which are missing in the formula (2.2). Another question concerns the validity of the independent pair approximation in the description of a collective effect which causes a transition. And at last, why does the effective potential between particles of the medium behave as the effective potential between infinitesimal external charges (with the same dielectric constant)? Indeed, a straightforward combination of (2.1) and (2.2) leads to

$$C(r) \underset{r \to \infty}{\sim} \frac{const}{r^{\Gamma/\varepsilon}}, \tag{2.4}$$

while the correlation between two infinitesimal external charges q_1 and q_2 behaves similarly as $r^{\beta q_1 q_2/\varepsilon}$. All these questions have been addressed by using low-fugacity expansions, as described in the next section. Notice that, in (2.4), the symbol $\sim_{r \to \infty}$ denotes the leading asymptotic term in the large-r expansion of $C(r)$. In other words, the terms that are dropped out in (2.4) decay faster than $1/r^{\Gamma/\varepsilon}$ when $r \to \infty$. In part I, as well as in part II, this notation will be used with the same meaning.

3. RESUMMATIONS OF THE LOW-FUGACITY EXPANSIONS NEAR THE ZERO-DENSITY CRITICAL POINT

3.1. Method

We consider the system in the Neutral Grand Canonical ensemble, where the parameters are the inverse temperature β and the fugacity z. As shown by Speer,[20] every low-fugacity Mayer graph is finite when Γ is greater than 4. This result holds without any chain resummations of long-range contributions, which are necessary at high temperature.[11] In ref. 16, we start from the low-fugacity Mayer expansions of the particle and charge correlations. Through the relation (1.2) of the linear response theory which links the dielectric constant to the second moment of the charge correlation, we derive the low-fugacity expansion of the inverse dielectric constant. Then, we proceed to a term-by-term analysis of the quantities of interest in the limit when $\Gamma - 4$ goes to zero.

At the order z^2, the correlation between particles of the same sign vanishes, because only neutral systems appear in the grand canonical ensemble. Thus, the charge correlation reduces to the correlation between particles with opposite signs, and it reads

$$C^{(2)}(r) = -2e^2 \frac{z^2}{\sigma^4} \left(\frac{\sigma}{r}\right)^{\Gamma}. \tag{3.1}$$

The corresponding value of $(1/\varepsilon)^{(2)}$ at the order z^2 is obviously

$$\left(\frac{1}{\varepsilon}\right)^{(2)} = -2\pi^2 \Gamma \frac{z^2}{\Gamma - 4} \tag{3.2}$$

where we have used that $C^{(2)}(r)$, and more generally $C(r)$, vanish for $r < \sigma$. So, when $\Gamma - 4$ goes to zero, $(1/\varepsilon)^{(2)}$ diverges as

$$\left(\frac{1}{\varepsilon}\right)^{(2)} \sim -8\pi^2 z \frac{z}{\Gamma - 4} \tag{3.3}$$

The analysis at higher orders is much more cumbersome and is summarized in Sections 3.2 and 3.3.

3.2. Exact Analysis at the Order z^4

First, we consider the leading asymptotic behavior of the particle correlations at the order z^4. The most probable configurations are those where every particle is paired with another one that has the opposite sign, as shown in Fig. 2. The positions of two particles are fixed at points $\underline{0}$ and $\mathbf{r}$, while the positions of the other two particles must be integrated over. In these configurations, the total Coulomb potential between the charges contains a dipolar term associated with the dipoles carried by the neutral pairs. When expanding the Boltzmann factor with respect to this dipolar interaction, the linear term vanishes for symmetry reasons, and we find,

$$\rho^{T(4)}_{++} \sim \rho^{T(4)}_{+-} \sim \frac{z^4}{\sigma^4} \left(\frac{\sigma}{r}\right)^4 \frac{\Gamma^2}{4} \left[\frac{1}{\sigma^4} \int_{\sigma < t} d\mathbf{t}\, t^2 \left(\frac{\sigma}{t}\right)^{\Gamma}\right]^2 \tag{3.4}$$

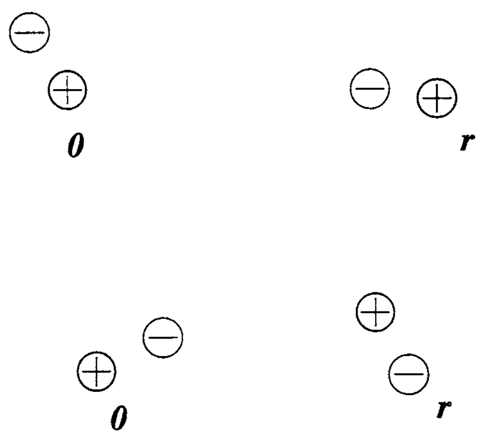

Fig. 2. The most probable configurations which contribute to the large distance behavior of $\rho^{T(4)}_{++}(\mathbf{0}, \mathbf{r})$ and $\rho^{T(4)}_{+-}(\mathbf{0}, \mathbf{r})$.

Thus, after integration over the orientation of the fluctuations of the relative position $\mathbf{t}$ of the particles inside a neutral pair, the particle correlations at large distances behave as the Boltzmann factor of a pair in the vacuum times the square of the dipolar interaction. We recall that the latter quantity is proportional to the polarizability of a single pair, which is finite when Γ is greater than 4.

A remarkable property is that the dipolar $1/r^4$-contributions (3.4) cancel out in the difference that appears in the charge correlation. As a consequence, $C^{(4)}(r)$ decreases faster than $1/r^4$. It can be shown that this asymptotic behavior is determined by the configurations shown in Fig. 3. In these configurations, the two particles whose positions must be

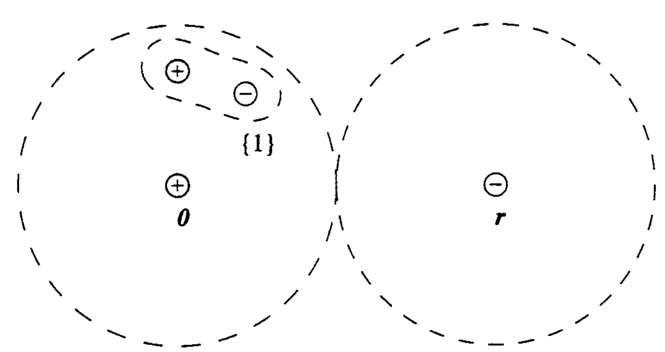

Fig. 3. The most probable configurations which contribute to the large distance behavior of $C^{(4)}(r)$. The integrated pair $\{1\}$ lies in the disks centered at $\mathbf{0}(\mathbf{r})$ with radius $r/2$.

integrated over form a pair with a size smaller than the distance between the two fixed particles. These configurations exist only in the correlation between particles with opposite signs.

An exact calculation shows that $C^{(4)}(r)$ can be rewritten as

$$C^{(4)}(r) = z^4 A(r; \Gamma) + z^4 R(r; \Gamma) \tag{3.5}$$

where

$$A(r; \Gamma) = \frac{1}{r^\Gamma}\left[\frac{a(\Gamma)}{\Gamma - 4}\ln\left(\frac{r}{\sigma}\right) + \frac{b(\Gamma)}{(\Gamma-4)^2} + \frac{c(\Gamma)}{(\Gamma-4)^2}\frac{1}{r^{\Gamma-4}}\right], \tag{3.6}$$

while R decays faster than $1/r^{2\Gamma-4}$. So A contains both the leading term and the first two subleading terms in the large-r behaviour of $C^{(4)}(r)$. The leading term behaves as $\ln r/r^\Gamma$, and the subleading terms are proportional to $1/r^\Gamma$ and $1/r^{2\Gamma-4}$. respectively. Moreover, the coefficients a, b and c remain finite when Γ goes to 4.

The decomposition (3.5) not only provides the large-r expansion of $C^{(4)}(r)$, but it is also useful for evaluating $(1/\varepsilon)^{(4)}$ in the limit where $\Gamma - 4$ goes to zero. Indeed, by using uniform bounds with respect to r and Γ, the contributions of $R(r; \Gamma)$ to the linear response expression (1.2) are found to diverge only as $1/(\Gamma-4)^2$. At the same time, the contributions of $A(r; \Gamma)$ diverge faster, namely as $1/(\Gamma-4)^3$. Thus, the most divergent contribution to $(1/\varepsilon)^{(4)}$ behaves as $z^4/(\Gamma-4)^3 = z(z/(\Gamma-4))^3$, and is of the same order as for $(1/\varepsilon)^{(2)}$ in the double limit where both z and $\Gamma - 4$ go to zero (see Eq. (3.3)). An important point is that this divergent contribution exactly arises from both the leading and subleading terms of $C^{(4)}(r)$ at large distances.

The present exact study of the z^4 terms suggests that the large-distance behavior of $C(r)$ and the behavior of $1/\varepsilon$ when $\Gamma - 4$ goes to zero are tightly related. This is illustrated in the following section which extends the present results to arbitrary orders z^{2n}.

3.3. Nested-Pair Mechanism at the Order z^{2n}

The large-distance behaviors of $\rho_{++}^{T(2n)}$ and $\rho_{+-}^{T(2n)}$ are obtained by a straightforward generalization of the previous calculation at the order z^4. Again, the configurations that contribute to the leading term are made of neutral clusters surrounding the two fixed particles. By using the harmonicity of the Coulomb potential and the symmetries, it is shown that the resulting correlations are proportional to the fluctuations of the dipolar interactions between the neutral clusters. Therefore, $\rho_{++}^{T(2n)}$ and $\rho_{+-}^{T(2n)}$ decay as $1/r^4$ with the same prefactor, which diverges when Γ goes to 4.

The $1/r^4$ algebraic tails cancel out in $C^{(2n)}$, as in $C^{(4)}$. In order to derive the large-r behavior of $C^{(2n)}$, it is most convenient to interpret the results at order z^4, in terms of a nested-pair mechanism. The part $C_\varepsilon^{(4)}$ of $C^{(4)}$ that gives the most divergent term in $1/\varepsilon$ coincides with the leading and subleading terms, $z^4 A(r;\Gamma)$, in the large-r expansion of $C^{(4)}$, and it can be written as

$$C_\varepsilon^{(4)}(r) = -2e^2 \frac{z^4}{\sigma^4} \left(\frac{\sigma}{r}\right)^\Gamma \int d\mathbf{x}\, d\mathbf{y}\, \mathcal{S}_{\mathcal{P}_0}(\mathcal{P}) \tag{3.7}$$

$C_\varepsilon^{(4)}$ appears as the product of the Boltzmann factor of a single pair $\mathcal{P}_0 = (\underset{\sim}{0}, \mathbf{r})$ in the vacuum times the integral of a screening factor $\mathcal{S}_{\mathcal{P}_0}(\mathcal{P})$ which describes how the pair of particles $\mathcal{P}_0 = (\underset{\sim}{0}, \mathbf{r})$ is screened by a smaller pair $\mathcal{P} = (\mathbf{x}, \mathbf{y})$ (See Fig. 3). We point out that the screening factor is nonzero only if the integrated pair is smaller than the nonintegrated pair.

This mechanism can be generalized at higher orders. We emphasize that we do not perform a complete analysis of the large-r behaviors and of their contributions to $1/\varepsilon$, contrarily to what was done in Section 3.2 for $C^{(4)}(r)$. For instance, (3.5) is an exact equation and the contribution of the rest $R(r;\Gamma)$ to $(1/\varepsilon)^{(4)}$ is controlled rigorously. In the following we will only determine the quantities of interest in the double limit $z \to 0$ and $\Gamma - 4 \to 0$, without a rigorous study of the discarded contributions. We introduce the part $C_\varepsilon^{(2n)}$ of the charge correlation that gives the most divergent contribution to $(1/\varepsilon)^{(2n)}$ near the critical point at zero density. $C_\varepsilon^{(2n)}$ coincides with the first terms in the large-r expansion of $C^{(2n)}$. The most divergent contribution to $(1/\varepsilon)^{(2n)}$ behaves as

$$z \times \left(\frac{z}{\Gamma - 4}\right)^{2n-1} \tag{3.8}$$

which is of the same order as $(1/\varepsilon)^{(2)}$ in the double limit where both z and $\Gamma - 4$ go to zero. This property is a signal of collective effects at critical points. Usually, these effects are dealt with through Renormalization Group approachs. In the field-theory method, which describes the Kosterlitz–Thouless transition by using the equivalence with the Sine–Gordon model, the collective effects are indeed handled by a double expansion in the fugacity and in $\Gamma - 4$.[18]

Similarly to $C_\varepsilon^{(4)}$, $C_\varepsilon^{(2n)}$ is entirely determined by nested pair configurations. Furthermore, for any given pair we need retain only the interaction

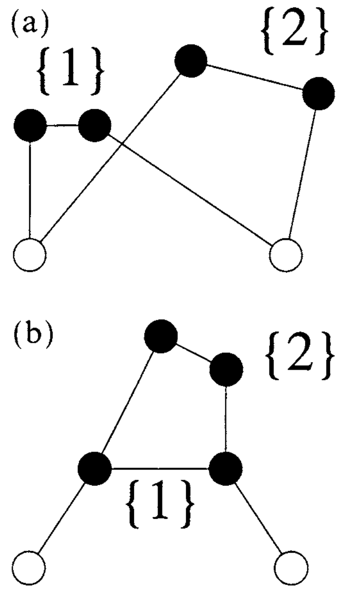

Fig. 4. (a) A configuration which contributes to $C_\varepsilon^{(6)}(r)$. The white circles denote the fixed neutral pair $\mathscr{P}_0$. The straight lines that connect the integrated pairs $\{1\}$ and $\{2\}$ to $\mathscr{P}_0$ represent the screening factor $\mathscr{S}$. (b) Another configuration which contributes to $C_\varepsilon^{(6)}(r)$ with the same notations as in Fig. 4a.

with the pair in which it is nested, and the latter interaction is described by the screening factor $\mathscr{S}$ (See (3.7)). For instance, $C_\varepsilon^{(6)}$ is the product of the Boltzmann factor of a single pair in the vacuum times an integral over the positions of four particles. The integrand is the sum of two contributions: a first one comes from configurations where the pair which is not integrated over is screened by two smaller pairs independently, whereas the second contribution arises from configurations where an integrated pair screens a larger integrated pair, which itself screens the nonintegrated pair. These configurations are shown in Fig. 4a and 4b, and the corresponding integrands respectively read

$$\left[\int d\mathbf{x}_1\, d\mathbf{y}_1\, \mathscr{S}_{\mathscr{P}_0}(\mathscr{P}_1)\right] \times \left[\int d\mathbf{x}_2\, d\mathbf{y}_2\, \mathscr{S}_{\mathscr{P}_0}(\mathscr{P}_2)\right] \tag{3.9}$$

and

$$\int d\mathbf{x}_1\, d\mathbf{y}_1\, \mathscr{S}_{\mathscr{P}_0}(\mathscr{P}_1) \int d\mathbf{x}_2\, d\mathbf{y}_2\, \mathscr{S}_{\mathscr{P}_1}(\mathscr{P}_2) \tag{3.10}$$

3.4. Resummations

Since every $\rho_{++}^{T(2n)}$ and $\rho_{+-}^{T(2n)}$ decays as $1/r^4$, the particle correlations ρ_{++}^{T} and ρ_{+-}^{T} are expected also to decay as $1/r^4$. The corresponding coefficient, which is the sum of the contributions from all orders in z, should remain finite in the dielectric phase near the zero density critical point. This perturbative result is corroborated by a recent rigorous analysis.[21] The three- and four-particle correlations can be also studied in a similar perturbative framework.[16] In particular, algebraic $1/r^2$-tails appear in the four-particle correlation when two neutral clusters are separated by a large distance r.

On the contrary, $C_\varepsilon^{(2n)}$ decays more and more slowly as n increases, namely as

$$\frac{1}{r^{\Gamma}}\left[\ln\left(\frac{r}{\sigma}\right)\right]^{n-1} \tag{3.11}$$

with a prefactor which diverges faster and faster, namely as $1/(\Gamma-4)^{n-1}$, when Γ goes to 4. Consequently, the coupled resummations for $C(r)$ and $1/\varepsilon$ are more cumbersome. In fact, the above nested-pair mechanism leads to a recurrence relation between $C_\varepsilon^{(2n)}$ and $C_\varepsilon^{(2p)}$, with $p \leqslant n-1$. This recurrence relation is equivalent to a system of coupled equations for $C_\varepsilon(r)$ and for an inverse dielectric constant which depends on the distance. This latter quantity, which naturally appears in the recurrence scheme, is defined as the truncated second moment of $C_\varepsilon(r)$. In fact, it turns out that this system is identical to the system of coupled equations (2.1)–(2.2) in the iterated mean-field model introduced by Kosterlitz and Thouless.

The full resummation of the z^2-expansion of $1/\varepsilon$ leads to

$$\frac{1}{\varepsilon} = 1 + \frac{(\Gamma-4)}{4}\left\{\left[1 - \frac{(8\pi z)^2}{(\Gamma-4)^2}\right]^{1/2} - 1\right\} \tag{3.12}$$

Notice that, at the order z^2, the r.h.s. of (3.12) reduces to the expression (3.3) of $(1/\varepsilon)^{(2)}$. The signal of the KT transition, which is characterized by the singularity of $1/\varepsilon$, coincides with the divergence of the series which gives $1/\varepsilon$ in terms of the parameter $z/(\Gamma-4)$. The corresponding radius of convergence is indeed given by Eq. (2.3). Moreover, the value of Γ/ε on the critical line is equal to 4, as conjectured in the literature.

3.5. Comments

According to the systematic low-fugacity survey, the charge correlation which appears in the phenomenological approach of Kosterlitz and Thouless must be identified with the part $C_\varepsilon(r)$ of the charge correlation $C(r)$ that gives the most divergent contributions to the inverse dielectric constant, near the zero-density critical point. Moreover, we point out that $C_\varepsilon(r)$ is the large-r behavior of $C(r)$ in this region, and reads

$$C_\varepsilon(r) = -2e^2 \frac{z^2}{\sigma^4}\left(\frac{\sigma}{r}\right)^{\Gamma/\varepsilon} A_0(z/(\Gamma-4))\left[1+\sum_{p=1}^{\infty} A_p(z/(\Gamma-4))\left(\frac{\sigma}{r}\right)^{p[(\Gamma/\varepsilon)-4]}\right] \tag{3.13}$$

Thus, the resummation scheme allows one to answer the questions presented in Section 2.2. Indeed, the leading term in (3.13) coincides with the mean-field prediction (2.4), and only the correlation between particles with opposite signs does contribute. This term has an exponent similar to that of the correlation between two external charges. We stress that this result was not obvious a priori, because the charge correlation involves internal (and not external) charges. Moreover, the relevant charge correlation (3.13) does not reduce to a single algebraic term. The nested-pair mechanism generates a series of algebraic subleading terms with increasing exponents, because $\Gamma/\varepsilon > 4$ in the dielectric phase. When the transition line is approached, all exponents collapse to 4 and all coefficients A_p vanish. There might appear marginal logarithmic terms on the critical line in agreement with the analysis of Høye and Olaussen.[19] In principle, the coefficients of the $1/r^4$ tails in ρ^T_{++} and ρ^T_{+-} should also vanish on the critical line. (The corresponding resummations with respect to the parameter $z/(\Gamma-4)$ remain to be carried out.)

4. PHASE DIAGRAM NEAR THE ZERO-DENSITY CRITICAL POINT

The equation of the KT critical line near the zero-density critical point has been exactly derived.[22] In fact, all theories give the equation (2.3) of the transition line in terms of the coupling constant Γ and the fugacity z. On the other hand, the fugacity expansions give

$$\rho \sim 2\pi\left(\frac{z^2}{\sigma^2}\right) \tag{4.1}$$

The combination of (2.3) and (4.1) leads to[22]

$$\Gamma_{KT}(\rho) - 4 \sim 4\sqrt{2\pi\rho\sigma^2} \tag{4.2}$$

On the transition line, $\Gamma - 4$ is proportional to the square root of the density, and the tangent to the critical line at $(\rho = 0, \Gamma = 4)$ is vertical in the plane temperature versus density. A similar result can be found in ref. 19 which deals with a system of charged soft disks. (The Coulomb potential is smoothly regularized at short distances $r < \sigma$.)

At higher densities, the resummation scheme is no longer valid. All the corresponding predictions have to be revisited. At the moment, numerical simulations[7] suggest that a first order transition between a conducting liquid and a dielectric gas occurs at some tricritical point. Recently, Fisher *et al.*[9] have studied finite-density effects by using the Bjerrum model, which is a mixture of free charges and neutral dipoles. In Fig. 5, we draw the phase diagram for charged hard disks according to most recent

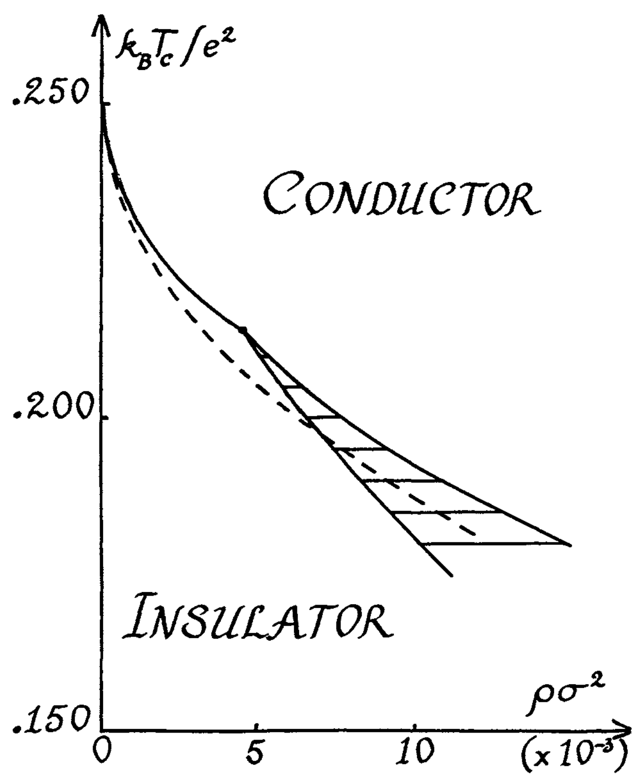

Fig. 5. The phase diagram in the plane $(\rho\sigma^2, k_B T/e^2)$. Dashed curve: the KT critical line extrapolated from the exact low-density behavior (3.15). Solid curve: the KT critical line and the liquid-gas coexistence curve computed in ref. 23. Hatched zone: the liquid-gas coexistence region. Circle: the tricritical point $(\rho_t\sigma^2 \simeq .00456, k_B T_t/e^2 \simeq .2138)$.

theoretical calculations.[23] The exact extrapolated low-density *KT* line (4.2) is close to the transition line incorporating finite-density effects.

Of course, there still remain some open questions from the point of view of first principles of Statistical Mechanics. First, another kind of resummation of low-fugacity expansions should be performed in the conducting phase near the zero-density critical point. This would provide an exact expression for the finite correlation length, which could be compared with the prediction of the phenomenological KT theory. In addition, it would be interesting to produce a derivation of the Bjerrum model according to first principles.

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